Use and Connect Mathematical Representations

> *Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.*

Effective mathematics teaching includes a strong focus on using varied mathematical representations. NCTM (2000) highlighted the important role of mathematical representations in the teaching and learning of mathematics by including the Process Standard for Representation in *Principles and Standards for School Mathematics*. Representations embody critical features of mathematical constructs and actions, such as drawing diagrams and using words to show and explain the meaning of fractions, ratios, or the operation of multiplication. When students learn to represent, discuss, and make connections among mathematical ideas in multiple forms, they demonstrate deeper mathematical understanding and enhanced problem-solving abilities (Fuson, Kalchman, and Bransford 2005; Lesh, Post, and Behr 1987).

Discussion

The general classification scheme for types of representations shown in figure 9 indicates important connections among contextual, visual, verbal, physical, and symbolic representational forms (Lesh, Post, and Behr 1987). Tripathi (2008) noted that using these "different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective that makes the picture (concept) richer and deeper" (p. 439). Students, especially young learners, also benefit from using physical objects or acting out processes during problem solving (National Research Council 2009).

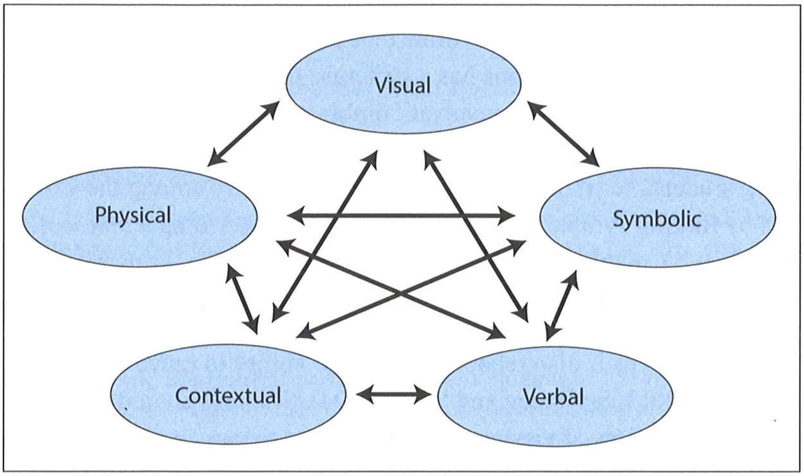


Fig. 9. Important connections among mathematical representations

According to the National Research Council (2001), "Because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas" (p. 94). The depth of understanding is related to the strength of connections among mathematical representations that students have internalized (Pape and Tchoshanov 2001; Webb, Boswinkel, and Dekker 2008). For example, students develop understanding of the meaning of the fraction $7/4$ (symbolic form) when they can see it as the quantity formed by "7 parts of size one-fourth" with a tape diagram or on a number line (visual form), or measure a string that has a length of 7-fourths yards (physical form).

Visual representations are of particular importance in the mathematics classroom, helping students to advance their understanding of mathematical concepts and procedures, make sense of problems, and engage in mathematical discourse (Arcavi 2003; Stylianou and Silver 2004). Visuals support problem solving as students consider relationships among quantities when they sketch diagrams or make tables and graphs. The visual representations also support discourse because the diagrams or drawings leave a trace of student problem solving that can be displayed, critiqued, and discussed. Math drawings and other visual supports are of particular importance for English language learners, learners with special needs, or struggling learners, because they allow more students to participate meaningfully in the mathematical discourse in the classroom (Fuson and Murata 2007). The visuals assist students in following the reasoning of their classmates and in giving voice to their own explanations as they gesture to parts of their math drawings and other visual representations.

Students' understanding is deepened through discussion of similarities among representations that reveal underlying mathematical structures or essential features of mathematical ideas that persist regardless of the form (Zimba 2011). For example, fractions are composed of the iteration of unit fractions, a structure that can be identified and discussed when students use paper strips as fraction models, draw tape diagrams or number lines, or use symbols. Likewise, the addition of fractions has a structure that is similar to that of the addition of whole numbers, in that all addition involves combining same-sized units, such as adding tens to tens or twelfths to twelfths. Mathematical structure can also be emphasized and discussed by asking students to translate or alternate directionality among the various representations, such as by linking symbols back to contexts (e.g., describing a real-world situation for 3×29 or $y = 3x + 5$), making a table of values for a given ratio, or making a graph based on the information in a table (Greeno and Hall 1997).

Success in solving problems is also related to students' ability to move flexibly among representations (Huinker 2013; Stylianou and Silver 2004). Students should be able to approach a problem from several points of view and be encouraged to switch among representations until they are able to understand the situation and proceed along a path that will lead them to a solution. This implies that students view representations as tools that they can use to help them solve problems, rather than as an end in themselves. If, by contrast, algebra tiles or base-ten blocks, for instance, are not used meaningfully, students may view use of the physical objects as the goal instead of reaching an understanding of how the tiles allow them to make sense of polynomials or how the base-ten blocks show the structure of the base-ten number system.

Illustration

Students' representational competence can be developed through instruction. Marshall, Superfine, and Canty (2010, p. 40) suggest three specific strategies:

1. Encourage purposeful selection of representations.
2. Engage in dialogue about explicit connections among representations.
3. Alternate the direction of the connections made among representations.

Consider the lesson presented in figure 10, and focus on how the teacher, Mr. Harris, uses these strategies with his third-grade students as they represent and solve a problem involving setting up chairs for a band concert.

The third-grade class is responsible for setting up the chairs for the spring band concert. In preparation, they have to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area.

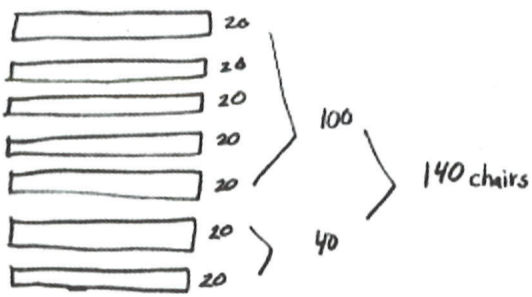
Mr. Harris explains to his students that they need to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. Next he asks the students to consider how they might represent the problem: "Before you begin working on the task, think about a representation you might want to use and why, and then turn and share your ideas with a partner."

The students then set to work on the task. Most sketch equal groups or decompose area models. Two students cut an array out of grid paper. A few students make a table or T-chart, listing the number of rows with the corresponding number of chairs. Some students use symbolic approaches, such as repeated addition or partial products.

A few students change representations as they work. Dominique starts to draw tally marks but then switches to using a table. When Mr. Harris asks her why, she explains that she got tired of making all those marks. Similarly, Jamal starts to build an array with connecting cubes but then switches to drawing an array. These initial attempts are valuable, if not essential, in helping each of these students make sense of the situation.

As the students work, the teacher poses purposeful questions to press them to consider critical features of their representations: "How does your drawing show 7 groups?" "Why are you adding all those twenties?" "How many twenties are you adding, and why?"

Before holding a whole-class discussion, Mr. Harris has the students find a classmate who used a different representation and directs them to take turns explaining and comparing their work as well as their solutions. For example, Jasmine, who drew the diagram shown below on the left, compares her work with Kenneth, who used equations, as shown on the right. Then Mr. Harris has the students repeat the process, finding another classmate and holding another share-and-compare session.



Jasmine's drawing

$20+20+20+20+20+20+20$

$40+40=80$

$80+20=100$

$100+20=120$

$120+20=140$

140 chairs

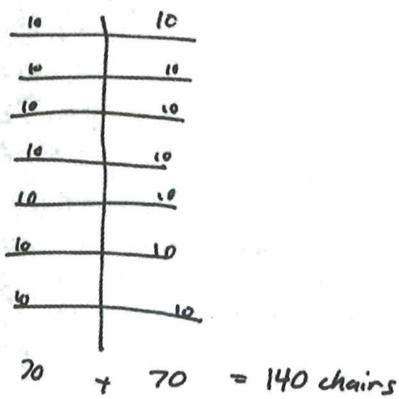
Kenneth's equations

Mr. Harris begins the whole-class discussion by summarizing the goal for the lesson as understanding how the different representations are related to the operation of multiplication. He first asks students to identify and explain how different visual representations show both the number of equal groups and the amount in each group as a structure of

Fig. 10. A third-grade lesson emphasizing mathematical representations to solve a task on setting up chairs for a band concert

multiplication. This prompts the students to compare diagrams with equal groups, arrays, and area models and discuss how they are similar and different. The students comment that it is easy to see the number of chairs in each row in some of the diagrams but not in others. Mr. Harris then writes 7 × 20 on the board and asks the students to explain how the expression matches each of the diagrams.

Finally, Mr. Harris has the students discuss and compare the representations of those students who considered the aisle and worked with tens rather than with twenties, such as Amanda, whose work is shown below. He asks them to take this final step, knowing that this informal experience and discussion of the distributive property will be important in subsequent lessons.



Amanda's work with tens

Fig. 10. *Continued*

Mr. Harris selects the task about the chairs for the band concert to focus on a problem situation that can be represented with arrays. The goal for the lesson is for students to understand how the structure of multiplication is evident within and among different representations. He chooses the numbers purposefully to build his students' conceptual understanding of multiplying one-digit whole numbers by multiples of 10, using strategies based on place value and properties of operations. He allows students to select and discuss their choices to represent the problem situation. Mr. Harris pays close attention to what students are doing, and the questions that he poses as they work and during the whole-class discussion help his students make explicit connections among the representations in ways that further their understanding of the central mathematical ideas of the lesson.

Teacher and student actions

Effective teaching emphasizes using and making connections among mathematical representations to deepen student understanding of concepts and procedures, support mathematical discourse among students, and serve as tools for solving problems. As students use and make connections among contextual, physical, visual, verbal, and symbolic representations, they grow in their appreciation of mathematics as a unified, coherent discipline. The teacher and student actions listed in the table below provide a summary of what teachers and students do in using mathematical representations in teaching and learning mathematics.

Use and connect mathematical representations **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Selecting tasks that allow students to decide which representations to use in making sense of the problems. Allocating substantial instructional time for students to use, discuss, and make connections among representations. Introducing forms of representations that can be useful to students. Asking students to make math drawings or use other visual supports to explain and justify their reasoning. Focusing students' attention on the structure or essential features of mathematical ideas that appear, regardless of the representation. Designing ways to elicit and assess students' abilities to use representations meaningfully to solve problems.	Using multiple forms of representations to make sense of and understand mathematics. Describing and justifying their mathematical understanding and reasoning with drawings, diagrams, and other representations. Making choices about which forms of representations to use as tools for solving problems. Sketching diagrams to make sense of problem situations. Contextualizing mathematical ideas by connecting them to real-world situations. Considering the advantages or suitability of using various representations when solving problems.