**A11  DEDUCE TURNING POINTS OF QUADRATIC FUNCTION BY COMPLETING THE SQUARE (higher tier)**

You should already be able to express a quadratic equation in the form *a*(*x + b*)2 + *c* by completing the square.

e.g. *x*2 − 6*x* + 3 = (*x* − 3)2 − 9 + 3 = (*x* − 3)2 − 6

e.g. 3*x*2 + 6*x* + 5 = 3[*x*2 + 2*x*] + 5 = 3[(*x* + 1)2 − 1] + 5 = 3(*x* + 1)2 + 2

We are now going to deduce the turning points of a quadratic function after completing the square.

**EXAMPLE 1**

Given *y* = *x*2 + 6*x* − 5, by writing it in the form *y* =(*x + a*)2 + *b*, where *a* and *b* are integers, write down the coordinates of the turning point of the curve. Hence sketch the curve.

*y* = *x*2 + 6*x* − 5

= (*x* + 3)2 − 9 − 5 Remember to halve the coefficient of *x*

 = (*x* + 3)2 − 14 and subtract (−3)2 to compensate

The turning point occurs when (*x* + 3)2 = 0, i.e. when *x* = −3

When *x* = −3, *y* = (−3 + 3)2 − 14 = 0 − 14 = −14

So the coordinates of the turning point is (−3, −14)

The graph *y* = *x*2 + 6*x* − 5 cuts the *y*-axis when *x* = 0, i.e. *y* = −5

Sketch:

 *y*

 *x*

 (0, −5)

 (−3, −14)

 When *y* = (*x + a*)2 + *b* then the coordinates of the turning point is (−*a*, *b*).

 The minimum or maximum value of *y* is *b*.

**EXAMPLE 2**

Given that the minimum turning point of a quadratic curve is (1, −6), find an equation of the curve in the form *y = x*2 *+ ax + b*.  Hence sketch the curve.

*y* = (*x* − 1)2 − 6 If the minimum is when *x* = 1, we know we have (*x* − 1)2

 = (*x*2 − *x* − *x* + 4) − 6 If the minimum is when *y* = −6, we know we have (*...*)2 − 6

 = *x*2 − 2*x* − 2

An equation of the curve is *y = x*2 − 2*x* − 2

The graph cuts the *y*-axis when *x* = 0, i.e. at *y* = −2

Sketch: It is a minimum turning point so the shape is

 *y*

 *x*

 (0, −2)

 (1, −6)

**NOTE:** There are other possible equations as, for example *y* = 4(*x* − 1)2 − 6 also has a turning point of (1, −6). If it was a maximum turning point then the coefficient of *x*2 would be negative.

**EXAMPLE 3**

Find the maximum value of −*x*2 + 4*x* − 7 and sketch the curve.

− *x*2 + 4*x* − 7 = − ( *x*2 − 4*x* + 7) First take out the minus sign

= − [(*x* − 2)2 − 4 + 7] Remember to use square brackets

= − [(*x* − 2)2 + 3]

= − (*x* − 2)2 − 3 Multiply (*x* − 2)2 and +3 by −1

The maximum value is −3

 *y* It is a maximum value so the shape is

*x*

(2, −3)

 (0, −7)

**EXERCISE:**

**1.** By writing the following in the form *y* = (*x + a*)2 + *b*, where *a* and *b* are integers, write down the coordinates of the turning point of the curve. Hence sketch the curve.

(a) *y* = *x*2 − 8*x* + 20 (b) *y* = *x*2 − 10*x* − 1

(c) *y* = *x*2 + 4*x* − 6 (d) *y* = 2*x*2 − 12*x* + 8

(e) *y* = −*x*2 + 6*x* +10 (f) *y* = 5 − 2*x* − *x*2

**2.** Given the following minimum turning points of quadratic curves, find an equation of the curve in the form *y = x*2 *+ ax + b*. Hence sketch each curve.

(a) (2, −3) (b) (−4, 1)

(c) (−1, 5) (d) (3, −12)

 (e) (1, −7) (f) (−4, −1)

**3.** Find the maximum or minimum value of the following curves and sketch each curve.

(a) *y* = *x*2 + 4*x* + 2 (b) *y* = 1 − 6*x* − *x*2

(c) *y* = −*x*2 + 2*x* − 3 (d) *y* = *x*2 − 8*x* + 8

 (e) *y* = *x*2 − 3*x* − 1 (f) *y* = −3*x*2 + 12*x* − 9

**4.** The expression *x*2 – 3*x* + 8 can be written in the form (*x* – *a*)2 + *b* for all values of *x*.

(i) Find the value of *a* and the value of *b*.

The equation of a curve is *y* = f(*x*) where f(*x*) = *x*2 – 3*x* + 8

The diagram shows part of a sketch of the graph of *y* = f(*x*).



The minimum point of the curve is *M*.

(ii) Write down the coordinates of *M*.

**5.** (i) Sketch the graph of f(*x*) = *x*2 – 6*x* + 10, showing the coordinates of the turning point and

 the coordinates of any intercepts with the coordinate axes.

(ii) Hence, or otherwise, determine whether f(*x*) − 3= 0 has any real roots.

 Give reasons for your answer.

**\*6.** The minimum point of a quadratic curve is (1, −4). The curve cuts the *y*-axis at −1.

 Show that the equation of the curve is *y* = 3*x*2 − 6*x* −1

**\*7.** The maximum point of a quadratic curve is (−2, −5). The curve cuts the *y*-axis at −13.

 Find the equation of the curve. Give your answer in the form *ax*2 + *bx + c*.

\* = extension

**ANSWERS:**

**1.** (a) (4, 4) (b) (5, −26)

(c) (−2, −10) (d) (3, −10)

 (e) (3, 1) (f) (2, −3)

**2.** (a) *y* = *x*2 − 4*x* + 1 (b) *y* = *x*2 + 8*x* +17

(c) *y* = *x*2 + 2*x* + 6 (d) *y* = *x*2 − 6*x* − 3

(e) *y* = *x*2 − 2*x* − 6 (f) *y* = *x*2 − 8*x* + 15

**3.** (a) minimum (−2, + 6) (b) maximum (−3, 10)

(c) maximum (1, −2) (d) minimum (4, −8)

 (e) minimum (1.5, −3.25) (f) maximum (−2, 3)

**4.** (i) *a* = 1.5 *b* = 5.75

 (ii) (1.5, 5.75)

**5.** (i) *y* f(*x*)

 10

 (3, 1)

 *x*

 (ii) It has 2 real roots as if you move the graph 3 down it will cut the *x*-axis twice

 as the minimum point will be (3, −2)

**6.** Minimum point is (1, −4) thus *y* = *A*(*x* − 1)2 − 4 = *Ax*2 − 2*Ax* + *A* − 4

 Cuts *y*-axis at −1, thus *A* − 4 = −1

 *A* = 3 *y* = 3*x*2 − 2(3)*x* + 3 − 4

 *y* = 3*x*2 − 6*x* −1

**7.** *y* = −2*x*2 − 8*x* − 13